THE ROBUSTNESS OF GAME DYNAMICS UNDER RANDOM PERTURBATIONS

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Cauvin & Mertikopoulos, ***A unified approach to the convergence of stochastic imitation dynamics,*** working paper, 2023

Motivation : the replicator dynamics

Notations :

- Consider a **species** or **population** of individuals
- **Genotypes** : $\mathcal{A} = \{1, \dots, A\}$
- $z_\alpha(t) =$ **number** of individuals of genotype α at time t.
- $x_\alpha(t) = z_\alpha / \sum_\beta z_\beta =$ **proportion** of genotype α at time t.
- $v_\alpha(x) =$ **fitness** of genotype $\alpha =$ offspring per individual of genotype α.

Evolution of populations : $\dot{z}_\alpha = z_\alpha v_\alpha$

link with Lokta-Volterra equations

Replicator Dynamics (Taylor & Jonker, 1978)

$$\dot{x}_\alpha = x_\alpha \left[v_\alpha - \sum_{\beta \in \mathcal{A}} x_\beta v_\beta \right] \tag{RD}$$

What is the impact of random perturbations on this dynamics ?

Outline

1. A Primer on Stochastic Analysis
2. Evolutionary Games & Dynamics
3. Stochastic Imitation Dynamics
4. Long-Run Behavior
5. Further Topics

Adding noise in differential equations

- **Goal** : give sense to
$$\dot{X}(t) = b(X(t)) + \sigma(X(t)) \cdot \text{“noise”}$$
- **Issue** : “noise” differentiable **almost nowhere** $\Longrightarrow$ not Lipschitz

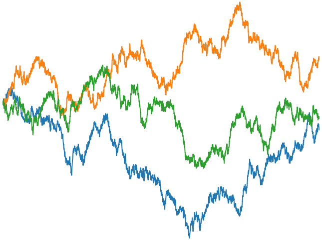

Figure. Typical trajectories of a natural ”noise” process.

How can we extend differential equations to deal with non-differentiable processes ?

Stochastic integrals & Stochastic differential equations (SDEs)

Itô's idea (Itô, 1944) :

- **Stochastic integral** :

$$\int_0^t f(s,\omega)\, dW(s) \text{ “=” } \lim_{\Delta t_k \to 0} \sum_k f(t_k,\omega)[W(t_{k+1}) - W(t_k)]$$

$W(t)$ is a random process called Brownian motion.

- **Stochastic equation (integral form)** :

$$X(t) = X(0) + \int_0^t b(X(s))\, ds + \int_0^t \sigma(X(s))\, dW(s)$$

- **Stochastic equation (differential form)** :

$$dX(t) = b(X)dt + \sigma(X)dW(t)$$

☞ Rigorous constructions can be found in (Øskendal, 2007), (Kuo, 2006) and (Karatzas & Shreve, 1998).

Stochastic calculus

A technical definition but easy computations :

- **Itô's formula** or the stochastic chain rule (simplified version) :

$$df(X) = f'(X)dX(t) + \frac{1}{2}f''(X)dX(t)\cdot dX(t)$$

rigorously, $dX(t)\cdot dX(t) \equiv d[X]_t$ with $[X]_t$ the quadratic variation

- Use formal **product rule** :

$$\begin{cases} dt\cdot dt & = 0 \\ dW(t)\cdot dt = dt\cdot dW(t) & = 0 \\ dW(t)\cdot dW(t) & = dt \end{cases}$$

- Example :

$$d\left[W(t)^2\right] = 2W(t)dW(t) + dt \implies \int_0^t W(s)dW(s) = \frac{1}{2}W(t)^2 - \frac{1}{2}t$$

A stochastic version of replicator dynamics

Biological derivation

- **Deterministic** evolution dynamics : $\dot{z}_\alpha = z_\alpha v_\alpha$
- **Perturbed** evolution dynamics : $dZ_\alpha = Z_\alpha(v_\alpha dt + \sigma_\alpha dW_\alpha)$ # aggregate shocks of nature on the fitness
- Use the (multidimensional) **Itô's formula** with $f(z) = z_\alpha / \sum_\beta z_\beta$ to obtain SDE verified by X_α

Replicator Dynamics with Aggregate Shocks (Fudenberg & Harris, 1992)

$$dX_\alpha = X_\alpha\Big[v_\alpha - \sum\nolimits_\beta X_\beta v_\beta\Big]dt + X_\alpha\Big[\sigma_\alpha dW_\alpha - \sum\nolimits_\beta X_\beta \sigma_\beta dW_\beta\Big]$$
$$- X_\alpha\Big[\sigma_\alpha^2 X_\alpha - \sum\nolimits_\beta \sigma_\beta^2 X_\beta^2\Big]dt$$

Different ways to add noise in replicator dynamics

- **Replicator Dynamics with Aggregate Shocks** (Fudenberg & Harris, 1992) :

$$dX_\alpha = X_\alpha\left[v_\alpha - \sum_\beta X_\beta v_\beta\right]dt + X_\alpha\left[\sigma_\alpha dW_\alpha - \sum_\beta X_\beta \sigma_\beta dW_\beta\right] - X_\alpha\left[\sigma_\alpha^2 X_\alpha - \sum_\beta \sigma_\beta^2 X_\beta^2\right]dt$$

- **Stochastic Exponential Learning** (Mertikopoulos & Moustakas, 2010) : # link with continuous-time FTRL

$$dX_\alpha = X_\alpha\left[v_\alpha - \sum_\beta X_\beta v_\beta\right]dt + X_\alpha\left[\sigma_\alpha dW_\alpha - \sum_\beta X_\beta \sigma_\beta dW_\beta\right] + \frac{1}{2}X_\alpha\left[\sigma_\alpha^2(1-2X_\alpha) - \sum_\beta \sigma_\beta^2 X_\beta(1-2X_\beta)\right]dt$$

- **Replicator Dynamics with Payoff Shocks** (Mertikopoulos & Viossat, 2016) :

$$dX_\alpha = X_\alpha\left[v_\alpha - \sum_\beta X_\beta v_\beta\right]dt + X_\alpha\left[\sigma_\alpha dW_\alpha - \sum_\beta X_\beta \sigma_\beta dW_\beta\right]$$

Our goal is to propose a general framework to unify the study of such stochastic dynamics.

Outline

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Non-atomic evolutionary games

Notations :

- Consider a **non-atomic population** # assumed to be of mass 1
- **Pure actions** or **genotypes**: $\mathcal{A} = \{1, \dots, A\}$
- **Mass of individuals** playing action α : $x_\alpha \in [0, 1]$ # also called a **mixed strategy**
- **Population state** : $x = (x_\alpha)_{\alpha \in \mathcal{A}} \in \Delta(\mathcal{A})$ # $\operatorname{supp} x = \{\alpha \in \mathcal{A} : x_\alpha > 0\}$
- **State space** : $\mathcal{X} \equiv \Delta(\mathcal{A})$
- **Payoff vector** : $v(x) \equiv (v_\alpha(x))_{\alpha \in \mathcal{A}}$ with $v_\alpha : \mathcal{X} \to \mathbb{R}$ Lipschitz

Population game

$$\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v) \equiv (\mathcal{A}, v)$$

☞ Here the game is defined in a single population for simplicity, but it can be easily extended to a multi-population setting

Game dynamics by revision

Idea : each (non-atomic) individual can revise their strategies periodically

- $dx_{\alpha\beta}$ = mass of individuals changing from strategy α to β over dt
- **Evolution of strategies** : $dx_\alpha = \text{inflow}(dt) - \text{outflow}(dt) = \sum_\beta dx_{\beta\alpha} - \sum_\beta dx_{\alpha\beta}$
- **Conditional switch rate** : $\rho_{\alpha\beta}(v(x), x)$ = probability of switching from strategy α to β
  $\implies dx_{\alpha\beta} \approx x_\alpha \rho_{\alpha\beta}\, dt$

Revision protocol dynamics

$$\dot{x}_\alpha = \sum_\beta x_\beta \rho_{\beta\alpha} - x_\alpha \sum_\beta \rho_{\alpha\beta}$$

☞ See (Sandholm, 2010) for a more extensive exploration of these dynamics and their derivations.

Imitation dynamics

Imitation revision protocols :

- $\rho_{\alpha\beta}(v,x) = x_\beta r_{\alpha\beta}(v,x)$ # imitates strategy of uniformly chosen opponent with conditional probability $r_{\alpha\beta}$
- $r_{\alpha\beta}$ = **conditional imitation rates** assumed Lipschitz
- **Monotone condition** : $v_\alpha < v_\beta \iff r_{\gamma\alpha} - r_{\alpha\gamma} < r_{\gamma\beta} - r_{\beta\gamma}$

  # needed to avoid non-rational behaviors, see (Mertikopoulos & Viossat, 2022)

Monotone Imitation Dynamics

$$\dot{x}_\alpha = x_\alpha \sum\nolimits_\beta x_\beta [r_{\beta\alpha} - r_{\alpha\beta}] \tag{ID}$$

Example (Replicator dynamics as imitation)

The replicator dynamics (RD) can be recovered from :

- (imitation of success) $r_{\alpha\beta} = K + v_\beta$;
- (imitation driven by dissatisfaction) $r_{\alpha\beta} = K - v_\alpha$;
- (pairwise imitation) $r_{\alpha\beta} = [v_\beta - v_\alpha]_+$.

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Adding noise in imitation dynamics

How to propose a perturbed version of (ID) **which has a "physical" meaning ?**

- **Solution** : approximate incertitude on payoffs by noise on conditional rates $r_{\alpha\beta}$
- Formally, $\hat{r}_{\alpha\beta} = r_{\alpha\beta} + \sigma_{\alpha\beta} dW_{\alpha\beta}$ on an infinitesimal time interval dt # $\sigma_{\alpha\beta} : \mathcal{X} \to \mathbb{R}$ assumed Lipschitz
- **Fully correlated noise** : $dW_{\alpha\beta} \cdot dW_{\alpha'\beta'} = C(\alpha, \beta; \alpha', \beta')dt$ # $\Leftrightarrow \operatorname{Cor}(W_{\alpha\beta}, W_{\alpha'\beta'}) = C(\alpha, \beta; \alpha', \beta')$

Stochastic Imitation Dynamics (Cauvin & Mertikopoulos, 2023)

$$dX_\alpha = X_\alpha \sum_\beta [r_{\beta\alpha} - r_{\alpha\beta}]dt + X_\alpha \sum_\beta X_\beta [\sigma_{\beta\alpha}(X) dW_{\beta\alpha} - \sigma_{\alpha\beta}(X) dW_{\alpha\beta}] \quad \text{(SID)}$$

$$= X_\alpha \sum_\beta [r_{\beta\alpha} - r_{\alpha\beta}]dt + X_\alpha \sigma_\alpha^T(X) dW_\alpha \quad \text{(SIDc)}$$

Examples

Independent source noise :

- If $r_{\alpha\beta}$ only depends on α, we can assume $\sigma_{\alpha\beta} = \sigma^s_\alpha$ and $W_{\alpha\beta} = W^s_\alpha$:

$$dX_\alpha = X_\alpha \sum\nolimits_\beta X_\beta [r_{\beta\alpha} - r_{\alpha\beta}]dt - X_\alpha \Big[\sigma^s_\alpha dW^s_\alpha - \sum\nolimits_\beta X_\beta \sigma^s_\beta dW^s_\beta\Big]$$

Independent target noise :

- If $r_{\alpha\beta}$ only depends on β, we can assume $\sigma_{\alpha\beta} = \sigma^t_\beta$ and $W_{\alpha\beta} = W^t_\beta$:

$$dX_\alpha = X_\alpha \sum\nolimits_\beta X_\beta [r_{\beta\alpha} - r_{\alpha\beta}]dt + X_\alpha \Big[\sigma^t_\alpha dW^t_\alpha - \sum\nolimits_\beta X_\beta \sigma^t_\beta dW^t_\beta\Big]$$

Example (Replicator dynamics with payoff shocks)

Taking $r_{\alpha\beta} = K + v_\beta$ and independent target noise, we recover the dynamics of Mertikopoulos & Viossat (2016) :

$$dX_\alpha = X_\alpha \Big[v_\alpha - \sum\nolimits_\beta X_\beta v_\beta\Big]dt + X_\alpha \Big[\sigma_\alpha dW_\alpha - \sum\nolimits_\beta X_\beta \sigma_\beta dW_\beta\Big]$$

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Dominated strategies

What are "bad places" for the dynamics ?

Dominated strategies :

- **Pure by pure** : $\alpha \in \mathcal{A}$ is **dominated** by $\beta \in \mathcal{A}$ if $v_\alpha(x) < v_\beta(x)$ for all $x \in \mathcal{X}$
- **Mixed by mixed** : $p \in \mathcal{X}$ is **dominated** by $p' \in \mathcal{X}$ if $\langle v(x), p \rangle < \langle v(x), p' \rangle$ for all $x \in \mathcal{X}$

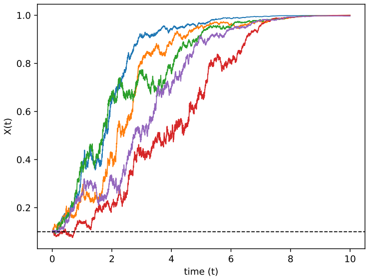
Rational behavior :

- Dominated strategies should be less and less, i.e., should become **extinct** along the dynamics
- Indeed what was proved for deterministic imitation dynamics, see (Akin, 1980) and (Nachbar, 1990)

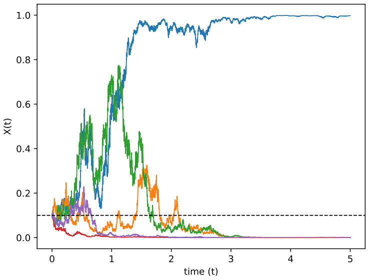
Extinction of dominated strategies

Definition (Extinction in stochastic regime)

1. $\alpha \in \mathcal{A}$ becomes **extinct** along $X(t)$ if $X_\alpha(t) \to 0$ (a.s.);
2. $p \in \mathcal{X}$ becomes **extinct** along $X(t)$ if $\min\{X_\alpha(t) : \alpha \in \operatorname{supp} p\} \to 0$ (a.s.).



(a) Low noise regime ($\sigma = 0.5$)



(b) High noise regime ($\sigma = 2$)

Figure. Trajectories of $X(t) \equiv X_\beta(t)$ for action space $\mathcal{A} = \{\alpha, \beta\}$ with α dominated by β.

Extinction of dominated strategies, cont'd

Perturbed payoff :

- Noise conditions will be given in term of the **perturbed payoff vector** $\tilde{v}$:

$$\tilde{v}_\alpha(x) = \sum\nolimits_\beta x_\beta [r_{\beta\alpha} - r_{\alpha\beta}] - \frac{1}{2}\sigma_\alpha^T(x)\widehat{C}(\alpha,\alpha)\sigma_\alpha(x) \qquad (\tilde{V})$$

Theorem (Cauvin & Mertikopoulos, 2023)

Let $X(t)$ be an interior solution orbit of (SID) and let $p \in \mathcal{X}$. Then p becomes extinct along $X(t)$ whenever there exists $p' \in \mathcal{X}$ such that

$$\langle \tilde{v}(x), p' - p \rangle > 0 \quad \text{for all } x \in \mathcal{X}.$$

- Does **not** require domination in original game $\mathcal{G}(\mathcal{A}, v)$
- Can be interpreted as p being dominated by p' in **perturbed game** $\widetilde{\mathcal{G}}(\mathcal{A}, \tilde{v})$
- Use dual process $Y_\alpha = \log X_\alpha$ and the fact that a stochastic integral $\int_0^t Z^T(s)dW(s)$ with Z bounded grows with rate $\mathcal{O}\left(\sqrt{t \log \log t}\right)$

Rates of extinction

Proposition (Cauvin & Mertikopoulos, 2023)

Let $X^*(t) = \min\{X_\alpha(t) : \alpha \in \operatorname{supp}(p)\}$ and $m = \inf_{x\in\mathcal{X}} \langle \tilde{v}(x), p' - p\rangle$. If p becomes extinct then :

1. **(asymptotic rate)** for all $\varepsilon > 0$ and t big enough,

$$X^*(t) \leq K \exp\Big\{-mt + 2(1+\varepsilon)\hat{\sigma}_p\sqrt{t\log\log t}\Big\} \quad (a.s.)$$

2. **(concentration inequality)** for all $\delta > 0$, $\# \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2}\, ds$

$$\mathbb{P}(X^*(t) > \delta) \leq \frac{1}{2}\operatorname{erfc}\left[\frac{1}{\sqrt{2}\hat{\eta}_p}\left(m\sqrt{t} - \frac{K_* - \log\delta}{\sqrt{t}}\right)\right]$$

3. **(hitting time)** for all $\delta > 0$, $\tau_\delta = \inf\{t \geq 0 : X^*(t) \leq \delta\}$ verifies

$$\mathbb{E}[\tau_\delta] \leq \frac{[K_* - \log\delta]_+}{m}$$

Main take away : convergence with rate $\propto e^{-t}$

Equilibria & Stochastic stability

Nash equilibria :

- $x^* \in \mathcal{X}$ is a **Nash equilibrium** of $\mathcal{G}$ if   # also equivalent to the Stampacchia variational inequality $\langle v(x^*), x - x^* \rangle \leq 0$

$$v_\alpha(x^*) \geq v_\beta(x^*) \quad \text{for all } \alpha \in \operatorname{supp}(x^*) \text{ and all } \beta \in \mathcal{A}$$

- x^* is a **strict Nash equilibrium** if inequality is strict for $\beta \notin \operatorname{supp}(x^*)$
- A strict Nash equilibrium is always **pure**, i.e., $\operatorname{supp}(x^*) = \{\alpha^*\}$

Definition (Stochastic stability of dynamical systems)

1. $x^* \in \mathcal{X}$ is **stochastically stable** if, for every $\varepsilon > 0$ and for every neighborhood $\mathcal{U}_0$ of x^*, there exists a neighborhood $\mathcal{U} \subseteq \mathcal{U}_0$ of x^* such that

$$X(0) \in \mathcal{U} \implies \mathbb{P}(X(t) \in \mathcal{U}_0 \text{ for all } t \geq 0) \geq 1 - \varepsilon$$

2. $x^* \in \mathcal{X}$ is **stochastically asymptotically stable** if it is stochastically stable and attracting : for every $\varepsilon > 0$ and for every neighborhood $\mathcal{U}_0$ of x^*, there exists a neighborhood $\mathcal{U} \subseteq \mathcal{U}_0$ of x^* such that

$$X(0) \in \mathcal{U} \implies \mathbb{P}\Big(X(t) \in \mathcal{U}_0 \text{ for all } t \geq 0 \text{ and } \lim_{t\to\infty} X(t) = x^*\Big) \geq 1 - \varepsilon$$

Perturbed equilibra

Perturbed equilibrium :

- $x^* \in \mathcal{X}$ verifies the **equilibrium condition** (EQ$^\sigma$) if

$$\tilde{v}_\alpha(x^*) \geq \tilde{v}_\beta(x^*) \quad \text{for all } \alpha \in \operatorname{supp}(x^*) \text{ and all } \beta \in \mathcal{A}, \tag{EQ$^\sigma$}$$

- x^* verifies **strictly** (EQ$^\sigma$) if inequality is strict for $\beta \notin \operatorname{supp}(x^*)$
- Condition (EQ$^\sigma$) can be interpreted as being a Nash equilibrium in the perturbed game $\widetilde{\mathcal{G}}$

Proposition (Cauvin & Mertikopoulos, 2023)

If x^* is a strict Nash equilibrium of $\mathcal{G}$, then it verifies strictly (EQ$^\sigma$).

- This result is a very surprising property of stochastic imitation dynamics, for instance it is not true for the replicator dynamics with aggregate shocks (see Imhof, 2005 and Hofbauer & Imhof, 2009)
- ⚠ If x^* verifies (EQ$^\sigma$), it is not necessarily a Nash equilibrium of $\mathcal{G}$

Stability of equilibria

Theorem (Cauvin & Mertikopoulos, 2023)

Let $x^* \in \mathcal{X}$ and let $X(t)$ be an interior solution orbit of (SID).

1. If $\mathbb{P}(\lim_{t\to\infty} X(t) = x^*) > 0$, then x^* verifies (EQ$^\sigma$);
2. If x^* is stochastically stable, then it verifies (EQ$^\sigma$);
3. If x^* verifies strictly (EQ$^\sigma$), then it is stochastically asymptotically stable;

- In particular, if x^* is a strict Nash equilibrium of $\mathcal{G}$, then it is stochastically asymptotically stable.

Rates of convergence

Proposition (Cauvin & Mertikopoulos, 2023)

Let α^* be a pure Nash equilibrium of $\mathcal{G}$ and $\mathcal{U}$ a neighborhood of x^* such that $\inf_{\alpha \neq \alpha^*,\, x \in \mathcal{U}} \tilde{v}_{\alpha^*}(x) - \tilde{v}_\alpha(x) > 0$.

Then, for all $\eta > 0$, there exists $\mathcal{U}_\eta \subset \mathcal{U}$ such that $X(0) \in \mathcal{U}_\eta$ implies :

1. **(asymptotic rate)** for all $\varepsilon > 0$ and t big enough,

$$X_{\alpha^*}(t) \geq 1 - K(A-1)\exp\left\{-mt + 2(1+\varepsilon)\sigma_* \sqrt{t \log\log t}\right\}$$

   with probability at least $1 - \eta$;

2. **(concentration inequality)** for all $\delta > 0$,

$$\mathbb{P}(X_{\alpha^*}(t) < 1 - \delta) \leq \eta + \frac{1}{2}(A-1)\operatorname{erfc}\left[\frac{1}{\sqrt{2}\hat{\eta}}\left(m\sqrt{t} - \frac{\tilde{K} - \log\left(\frac{\delta}{A-1}\right)}{\sqrt{t}}\right)\right];$$

3. **(hitting time)** for all $\delta > 0$, $\tau_\delta = \inf\{t \geq 0 : X_{\alpha^*}(t) \geq 1 - \delta\}$ verifies

$$\mathbb{E}[\tau_\delta \mathbb{1}\{X(t) \in \mathcal{U}\ \forall t \geq 0\}] \leq \frac{(A-1)\left[\tilde{K} - \log\left(\frac{\delta}{A-1}\right)\right]_+}{m}.$$

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Stochastic exponential dynamics

Exponential Dynamics

$$\dot{x}_\alpha = x_\alpha L_\alpha(x) \tag{ED}$$

Stochastic Exponential Dynamics

$$dX_\alpha = X_\alpha \hat{L}_\alpha(X)dt + X_\alpha \sigma_\alpha^T(X)dW_\alpha \tag{SED}$$

where $\hat{L}_\alpha = L_\alpha + I_\alpha$ with I_α a bias correction term.

Examples (A generalization of stochastic replicator dynamics)

The class of stochastic exponential dynamics contains :

- Stochastic imitation dynamics (SID) # even without monotone assumption
- Replicator dynamics with random mutations of Mertikopoulos & Viossat (2022)
- Replicator dynamics with aggregate shocks of Fudenberg & Harris (1992)
- Stochastic exponential learning of Mertikopoulos & Moustakas (2010)

Stochastic exponential dynamics, cont'd

Perturbed vector & equilibrium :

- Perturbed vector field $\tilde{L}$:

$$\tilde{L}_\alpha(x) = \hat{L}_\alpha(x) - \frac{1}{2}\sigma_\alpha^T(x) C(\alpha,\alpha)\sigma_\alpha(x)$$

- $x^* \in \mathcal{X}$ verifies the **equilibrium condition** (EQ$^\sigma_L$) if

$$\tilde{L}_\alpha(x^*) \geq \tilde{L}_\beta(x^*) \quad \text{for all } \alpha \in \operatorname{supp}(x^*) \text{ and all } \beta \in \mathcal{A}. \tag{EQ$^\sigma_L$}$$

- x^* verifies **strictly** (EQ$^\sigma_L$) if inequality is strict for $\beta \notin \operatorname{supp}(x^*)$

Theorem (Cauvin & Mertikopoulos, 2023)

Let $p, x^* \in \mathcal{X}$ and let $X(t)$ be an interior solution orbit of (SED).

1. If there exists $p' \in \mathcal{X}$ such that $\left\langle \tilde{L}(x), p' - p \right\rangle > 0$ for all $x \in \mathcal{X}$, then p becomes extinct along $X(t)$
2. If $\mathbb{P}(\lim_{t\to\infty} X(t) = x^*) > 0$, then x^* verifies (EQ$^\sigma_L$)
3. If x^* is stochastically stable, then it verifies (EQ$^\sigma_L$)
4. If x^* verifies strictly (EQ$^\sigma_L$), then it is stochastically asymptotically stable
5. ~~If x^* is a strict Nash equilibrium of $\mathcal{G}$, then it is stochastically asymptotically stable~~

What's more and what's next ?

- How to generalize proofs behind extinction and stability of stochastic dynamics, and extend it to more general sets ? $\longrightarrow$ introduce notion of **variationally aligned** sets, they are indeed (stochastically) stable and contain both dominated strategies and strict Nash equilibriums.

- Our work has link with **stochastic regularized learning** of Bravo & Mertikopoulos (2017) $\longrightarrow$ our results and rates also hold almost the same.

- Study of **stochastic mirror descent dynamics** over more complicated spaces than a simplex (e.g., convex polytopes, convex sets, manifolds, ...).

References I

[1] Akin, E. Domination or equilibrium *Mathematical Biosciences*, 50:239–250, 1980

[2] Bravo, M. and Mertikopoulos, P. On the robustness of learning in games with stochastically perturbed payoff observations. *Games and Economic Behavior*, 103(John Nash Memorial issue): 41–66, May 2017.

[3] Cauvin, P.-L. and Mertikopoulos, P. A unified approach to the convergence of stochastic imitation dynamics. working paper, 2023.

[4] Fudenberg, D. and Harris, C. Evolutionary dynamics with aggregate shocks. *Journal of Economic Theory*, 57(2):420–441, August 1992.

[5] Hofbauer, J. and Imhof, L. A. Time averages, recurrence and transience in the stochastic replicator dynamics. *The Annals of Applied Probability*, 19(4):1347–1368, 2009.

[6] Hofbauer, J. and Sigmund, K. Evolutionary game dynamics. *Bulletin of the American Mathematical Society*, 40(4):479–519, July 2003.

[7] Imhof, L. A. The long-run behavior of the stochastic replicator dynamics. *The Annals of Applied Probability*, 15(1B):1019–1045, 2005.

[8] Itô, K. Stochastic Integral. *Proceedings of the Imperial Academy of Tokyo*, 20:519–524, 1944.

[9] Karatzas, I. and Shreve S. E. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, Berlin, 1998.

[10] Khasminskii, R. Z. *Stochastic Stability of Differential Equations*. Number 66 in Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin, 2nd edition, 2012.

[11] Kuo, H. *Introduction to Stochastic Integration*. Springer-Verlag, Berlin, 2006.

[12] Mertikopoulos, P. and Moustakas, A. L. The emergence of rational behavior in the presence of stochastic perturbations. *The Annals of Applied Probability*, 20(4):1359–1388, July 2010.

[13] Mertikopoulos, P. and Viossat, Y. Imitation dynamics with payoff shocks. *International Journal of Game Theory*, 45(1-2):291–320, March 2016.

References II

[14] Mertikopoulos, P. and Viossat, Y. Survival of dominated strategies under imitation dynamics. *Journal of Dynamics and Games*, 9(4, William H. Sandholm memorial issue) :499–529, October 2022.

[15] Nachbar, J. Evolutionary selection dynamics in games: convergence and limit properties. *International Journal of Game Theory*, 19:59–89, 1990

[16] Øskendal, B. *Stochastic Differential Equations*. Springer-Verlas, Berlin, 6th edition, 2007.

[17] Sandholm, W. H. *Population Games and Evolutionary Dynamics*. MIT Press, Cambridge, MA, 2010.

[18] Taylor, P. D. and Jonker, L. B. Evolutionary stable strategies and game dynamics. *Mathematical Biosciences*, 40(1-2):145–156, 1978.